東海大學統計學系碩士班

碩士論文

指導教授：王榮琮博士

Valuation of Catastrophe Risk Derivatives by Jump Loss Processes

應用跳躍損失過程評價巨災風險衍生性金融商品

研究生：趙旋佑

中華民國一百零六年七月

東海大學碩士班研究生

論文口試委員審定書

統計學系碩士班趙旋佑君所提之論文

Valuation of Catastrophe Risk Derivatives by Jump Loss Processes

經本委員會審議，認為符合碩士資格標準。

論文口試委員召集人＿＿＿＿＿＿＿（簽章）

委員＿＿＿＿＿＿＿

＿＿＿＿＿＿＿

中華民國 106 年 07 月 10 日

Valuation of Catastrophe Risk Derivatives by Jump Loss Processes

Submitted for the Degree of Master of Science

Chao Hsuan Yu

Department of Statistics

Tunghai University

July 2017

Contents

應用跳躍損失過程評價巨災風險衍生性金融商品

摘要

本論文主要是探討巨災風險衍生性金融商品的定價。我們使用了雙重觸發機制的賣權來進行建模，模型取決於標的物價格及累計損失的程度。在風險中性定價假設下，雙重觸發賣權的價值可以透過折現期望值來表示。我們假設損失賠償的發生是服從非齊性卜松過程，而每次損失金額的大小則為獨立且分配相同的隨機變數。總體的損失可以由一個複合過程來表示，本論文針對賠償發生的過程考慮數個不同的 NHPP，對於損失金額的大小則使用幾個不同的隨機分布來分析。透過這些特徵，本論文建構了巨災賣權模型的定價，並進行實際數據分析，我們針對不同參數如何對選擇權價格的影響進行討論。

Valuation of Catastrophe Risk Derivatives by Jump Loss Processes

Abstract

This thesis studies the valuation of catastrophe risk derivatives. The price of catastrophe risk derivatives is modeled by a double trigger put option, which depends on the underlying asset price and the cumulative level of insurance loss. Under the risk-neutral pricing measure, the value of double trigger put option can be expressed as a discounted expectation. This expectation involves, apart from the usual Black-Scholes put option, a loss claims process and its corresponding aggregate loss. The loss claims are assumed to be non-homogeneous Poisson processes, and the sizes of loss claims are characterized by a sequence of independent and identically distributed random variables. The accumulated loss process can be described by a compound process. This thesis develops a model for catastrophe put option by considering several non-homogeneous Poisson processes for loss claims arrivals and different distribution functions for the sizes of losses. We analyze and validate the pricing model and discuss how different parameters affect the value of the catastrophe risk option.

1. Introduction

Catastrophe Risk Securitization:

Catastrophe risk events referring to both human and natural disasters are unpredictable. Human disasters, such as terrorist attacks, War, strike, airplane crash lead to significant impacts to our society. Natural disasters, such as earthquakes, tsunamis, typhoons, hurricanes, tornadoes, cold waves, heat waves, floods, droughts, earth flow can cause a tremendous loss of property. Losses caused by natural disasters are usually greater and much more difficult to predict than those caused by human factors. For example, in August 1992, Hurricane Andrew caused extreme damage to the southeastern part of the United States, and brought $26.5 billion in total losses. Hurricane Andrew is a catastrophe event based on the rules of Property Claim Service (PCS) in the United States, where a "catastrophe" is defined as a disaster with more than $25 million of property loss.

Losses from such a destructive disaster could severely affect insurers, policy holders, and the reinsurance companies. Traditionally, insurers purchase reinsurance contracts to hedge and transfer underwriting risk to reinsurance markets and capital markets such that loss can be reduced. However, insurers still suffered huge losses, and even have financial crisis. Therefore, the reinsurance companies, under financial pressure, either increase reinsurance transaction costs to cover catastrophic loss, or restrict further reinsurance conditions. Consequently, it is much harder for reinsurance companies to compensate the catastrophic losses at a reasonable price. In the past two decades, it is not easy to find a reinsurance company which can make up for catastrophe losses at a reasonable cost.

In order to expand the capacity of reinsurance industry, securitization of the accumulated catastrophic losses in financial markets has become a timely and desirable alternative to the traditional reinsurance norm (D'Arcy and France, 1992). Catastrophe risk securitization is a new risk management tool, and many commodities are insurance-linked financial securities. The concepts of these products involve multiple risks and multiple periods. The multiple risks involve the combination of various insurance risk (e.g., actuarial science) and financial risks (e.g., derivatives pricing). Extending single period to multiple periods is beneficial in more stable premiums (Cox et al., 2004).

Catastrophe derivative financial products generally include catastrophe futures, catastrophe bonds, and catastrophe options. Derivatives are financial contracts whose value is based on other underlying assets, that is, the value of derivatives is derived from the value of the underlying assets. Therefore, derivatives are also called *contingent claims*.

Catastrophe Futures:

The Chicago Board of Trade (CBOT) launched the Catastrophe Insurance Futures and Catastrophe Insurance Futures Options in 1992. The CBOT designed the loss ratio index, which was calculated by the Insurance Services Office (ISO) based on the loss data of more than 25 selected companies. The catastrophe loss ratio index is the ratio of the total amount of catastrophe loss per quarter L_t to the quarterly premium Π, that is, $\frac{L_t}{\Pi}$. The decision on the price of the insurance futures F_t is 25,000 US dollars per unit contract plus the catastrophe loss ratio (upper limit is 2):

$$F_t = 25{,}000 \times \mathrm{Min}(\frac{L_t}{\Pi}, 2)$$

In order to limit the unexpected loss of credit risk, the CBOT limits the maximum loss rate at 200%. However, there is no event that practically reaches the maximum loss rate, and we can thus ignore the maximum loss rate and write the value P_t of catastrophe insurance call option as

$$P_t = (F_t - K)_+ = \frac{25{,}000}{\Pi}(L_t - B)_+$$

where K is the strike price, $B = \frac{\Pi K}{25{,}000}$, and $(a-b)_+ = \max(a-b, 0)$.

Dassios and Jang (2003) substitute the total amount of money L_t by $C_t = \sum_{i=1}^{N_t} Z_i$, where $Z_i{}'$s is the claim amounts and N_t is the number of claims for time, and assume that the strength function is the shot noise of the Cox process. Ignoring the maximum loss ratio, Dassios and Jang (2003) used the piecewise deterministic Markov process theory to explore the pricing of the following insurance future

$$\frac{25{,}000}{\Pi} E[(C_t - B)_+].$$

Catastrophe Bonds:

Catastrophe risk bonds (CAT bonds) are one of the most important financial securities associated with insurance, also known as insurance-linked bonds. The first successful CAT bond was $ 85 million issued by Hanover Re in 1994 (Laster, 2001). Another CAT bond was issued by a non-financial firm, in 1999, which covered the earthquake losses in Tokyo for the company Oriental Land (Cummins, 2008). Since then insurers have increased to use catastrophe bonds to transfer insurance risks to capital markets. Catastrophe bonds have advanced into valuable risk management and investment tools by combining elements from both the reinsurance and debt capital markets.

Because the occurrence of catastrophe is largely unpredictable, valuation of CAT bonds is very difficult. In spite of this, the study of pricing bond models has played a crucial role in the prevention and mitigation of natural disasters. The structure of the CAT bond can be described by Figure 1 (Ma and Ma, 2013).

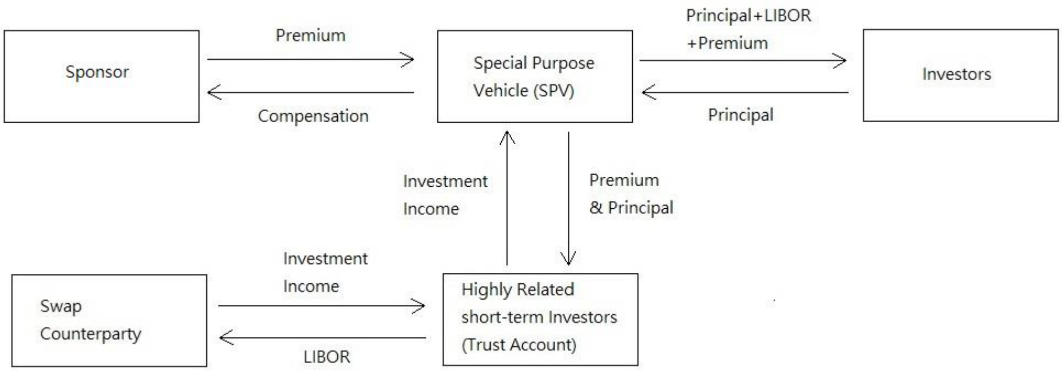

Figure 1. The structure of CAT bond

The CAT bound structure involves a sponsor (insurer, reinsurer, or government) who seeks to transfer the risk to investors who accept the risk for higher expected returns. The transfer of risk to the capital market is achieved by creating a special purpose vehicle (SPV) that provides coverage to the sponsor and issues independently regulated bonds to investors. The sponsor pays a premium in exchange for a pre-specified coverage if a catastrophic event of a certain magnitude takes place and investors purchase a bond. The SPV collects the capital and invests the proceeds in safe and short-term securities (e.g., Treasure bonds), which are held in a trust account. The returns generated from this trust account are usually swapped for floating returns based on the London interbank offered rate (LIBOR) that are supplied by a highly rated swap counterparty. The reason for the swap is to immunize the sponsor and the investors from interest rate (market-to-market) risk and default risk (Cummins, 2008).

If the covered event (also known as trigger event) does not happen during the term of the CAT bonds, investors receive their principal plus a compensation for the catastrophic risk exposure. However, if a catastrophic risk event occurs and triggers specified in the bond contract during the risk-exposure period, then the SPV compensates the sponsor according to the CAT bond contract. This results in a partial or full principal to the investors (Loubergé et al., 1999).

Obviously, defining triggering events plays an important role in structured CAT bonds. This catastrophe event should be measurable and easy to understand. Ma and Ma (2013) derived the pricing formula for the CAT bonds under the stochastic interest rate environment. They suggested two trigger scenarios

$$P_{CAT}(T) = \begin{cases} Z, & \text{if } L_T \leq D \\ pZ, & \text{if } L_T > D \end{cases}$$

and

$$P'_{CAT}(T) = \begin{cases} Z + C, & \text{if } L_T \leq D \\ Z, & \text{if } L_T > D \end{cases}$$’

where D is the threshold for the CAT bonds contracts, and L_T is the total amount of debt due to the maturity of the bond, when the time must be paid to the bondholder's principal ratio, for the debt.

Catastrophe Options:

Option contract is a derivative financial product with special profit and loss characteristics. After paying premium, buyer has the right to buy or sell a certain amount of underlying asset at a predetermined price during a period of time. After receiving premium, seller has the obligation to sell or buy the underlying asset when buyer exercises his right during the term of option.

A standard European put option provides the owner with a payment at time T of $K - S_T$ if $S_T < K$, since the option owner can buy the stock in the market for a price of S_T per share and immediately sell it under the terms of the put option for K per share. The option expires without a payment if $S_T \geq K$. A standard European put option is thus written as

$$(K - S_T)_+ = \begin{cases} \mathrm{K} - S_T, & if\ S_T < K \\ 0, & if\ S_T \geq K \end{cases}$$

where K is the exercise price, and S_T is the value of the underlying asset at time T. The client has the right to sell one share at time T for a price of K. The client will exercise the option only if the market price of equity is below the exercise price K.

As we don't know the future value of the underlying asset, the value of European put option is given by the expected discounted value

$$P_0{}^{(T)} = e^{-rT}E[(K - S_T)_+].$$

This formula involves several variables: the underlying asset price (S), the strike price (S_T), the maturity time (T), the volatility (σ), and the risk-free interest rate (r). Black and Scholes (1973) derived the value of European put option as

$$P_0{}^{(T)} = Ke^{-rT}\Phi\left(-d_1 + \sigma\sqrt{T}\right) - S_0\Phi(-d_1),$$

where S_0 is the current price of the underlying asset, Φ is the cumulative distribution function (cdf) of the standard normal distribution, and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

In 1996 the first catastrophe equity put option or CatEPut option was issued on behalf of RLI Corp., giving RLI the right to issue up to $50 million of cumulative convertible preferred shares (Punter, 2001). The CatEPut option gives RLI the right to issue convertible preferred shares at a fixed price, similar to the general call options. This right can only be exercised if the catastrophe cumulative loss exceeds the critical coverage limits during the term of the option. CatEPut option is a special case of the so-called double trigger option. A double trigger options depends on two random variables: the underlying asset price and the level of insurance loss. The payoff to the CatEPut option can be written as (e.g., Jaimungal and Wang, 2005)

$$I_{\{L_T - L_t > D\}}(K - S_T)_+ = \begin{cases} K - S_T, & S_T < K \text{ and } L_T - L_t > D \\ 0, & S_T \geq K \text{ or } L_T - L_t \leq D \end{cases}$$

where S_T is the market price of the asset underlying the option, $L_T - L_t$ is the overall claims during $[t, T)$, D is the loss level of the trigger, and K is the price at which the issuer is obliged to purchase the stock when the loss exceeds D.

Motivation of Study:

Pricing catastrophe risk derivatives is an important issue in reinsurance industry. In practice, many of existing pricing models assumed that catastrophe claim arrivals follows a homogeneous Poisson process (HPP) and applied compound Poisson process to aggregate the catastrophe asset loss. However, most models do not consider that catastrophe claim arrivals follow a non-homogeneous Poisson processes (NHPP). Since the loss claims often arrive in a particular trend or in cluster, therefore it is reasonable to assume that the catastrophe loss claims arrive according to NHPPs.

This thesis explores the pricing of European catastrophe put options, based on the risk neutral evaluation hypothesis. We apply the cumulative process for pricing European catastrophe put option, through various kinds of stock price jump models. Stock price jump models are characterized by NHPPs.

Contents:

The rest content of this thesis are given as follows. Section 2 gives the valuation theory, which provides the foundation for deriving the pricing model. Section 3 describes the double trigger model for catastrophe risk put options, including jump processes (NHPPs) and accumulated loss distributions. Pricing catastrophe risk put option is given in Section 4. The pricing model involves the accumulated loss distribution, the NHPP, and the Black-Scholes model. In order to evaluate the pricing model, in Section 5 we derive distributions for the size of loss claims and NHPPs for loss claims arrivals. In Section 6, a real data set is applied to fit both distributions and NHPPs in order to calibrate parameters of the pricing model. A numerical analysis is given in Section 7. Finally, Section 8 gives the conclusions.

2. Valuation Theory

Let $\{S_t, t \geq 0\}$ be the value of a risky underlying asset defined on a risk-neutral filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$ with $\{S_t, t \geq 0\}$ adapted to the filtration $(F_t)_{t\geq 0}$, where $F_t = \sigma\{S_u, 0 \leq u \leq t\}$ and P is a probability measure on $F \equiv \{F_t, t \geq 0\}$.

Consider an arbitrage-free financial market. Under the risk-neutral pricing measure Q (or an equivalent martingale measure), the value of the contingent claim $\{C_T, T > t\}$ at time t can be expressed by discounted expectations. Let V_t denote the value of the option at time *t*, then

$$V_t = E_t{}^Q[D(t,T)C_T|F_t]$$

where $E_t{}^Q$ denotes the expectation under the risk-neutral pricing measure Q, given F_t. The value $D(t,T) = \exp(-\int_t^T r(s)ds)$ is a stochastic discount factor (Cox et al., 2004). This formula V_t can be stated as that in an arbitrate-free financial market there exists a stochastic process $D(t,T)$ that prices the contingent claim C_T (e.g., see Ma and Ma, 2013).

If interest rates are deterministic, the discount factor $D(t,T)$ can be extracted from the expectation:

$$V_t = D(t,T)E_t{}^Q[C_T|F_t]$$

In this case, $D(t,T)$ is the value of a *T*-maturity zero-coupon bond at time *t*.

The most well-known put option is the Black-Scholes model where $D(t,T) = e^{-r(T-t)}$ and $C_T = (K - S_T)_+$. Another example is given by Cox et al. (2004) with $D(t,T) = e^{-r(T-t)}$ and $C_T = I_{\{N_t \geq n\}}(K - S_T)_+$, where N_t is a HPP with parameter λ, and n is a parameter on the contract.

In the presence of stochastic interest rates, a similar factorization can be obtained by performing a measure change to the forward-neutral measure Q^T (for details see, Jaimungal and Wang, 2005).

The Black-Scholes Model

The Black-Scholes model is the most famous financial model. In the Black-Scholes model, the price of the underlying asset, S_t , is defined on a risk-neutral filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$ and is governed by the following stochastic differential equation (SDE), the so-called geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\mu \neq 0$ is the mean return of the asset, σ is the volatility of return, and W_t is a standard Brownian motion under risk-neutral probability measure.

There is a unique solution for this SDE with initial value S_0. Using Itô's formula (see Dothan, 1990), the closed form solution is given by

$$S_t = S_0 \exp[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t]$$

where S_t is the price of the underlying asset at time *t*.

Let Q denote the new measure induced by the change of processes. Measure Q is equivalent to measure P and whose Randon-Nikodym derivative is given by (Dothan, 1990, p.210)

$$\frac{dQ}{dP} = \exp[-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T]$$

for $0 \leq t \leq T$. The equilibrium price measure Q exists and is finite. Define a new Brownian motion under Q

$$\widetilde{W_t} = W_t + \frac{\mu - r}{\sigma} t$$

so that we have *Q*-a.s.,

$$S_t = S_0 + r\int_0^t S_u du + \sigma \int_0^t S_u d\widetilde{W_u}, \qquad 0 \leq t \leq T$$

Hence we have Q-a.s

$$S_t = S_0 \exp[\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{W_t}]$$

where r is a risk-free rate.

The logarithm of the underlying asset S_t follows the normal distribution with mean $(r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$:

$$lnS_t \sim N\ (\left(r - \frac{1}{2}\sigma^2\right) t\ , \sigma^2 t)$$

The derivation of the closed form solution using the method of transformation is given in Appendix A.

The price of a European catastrophe put option with maturity *T* is the expected discounted value

$$V_t = E_t{}^Q [e^{-rT}(K - S_T)_+ | F_t]$$

Where $\{e^{-rt}S_t : t \geq 0\}$ is a *Q*-martingale.

3. Model Assumptions

In this section, assumptions for the proposed pricing model are given. They are double trigger option, loss claims process, and accumulated loss distribution.

3.1 Double Trigger Option

A double trigger option depends on the underlying asset price and the level of insurance loss. It can be activated only if both conditions are satisfied. Specifically,

$$I_{\{L_T - L_t > D\}}(K - S_T)_+ = \begin{cases} K - S_T, & S_T < K \ \ and \ \ L_T - L_t > D \\ 0, & S_T \geq K \ \ or \ \ L_T - L_t \leq D \end{cases}$$

where S_T is the market price of the asset underlying the option, $L_T - L_t$ is the overall claims during $[t,T)$, D is the level of the loss of the trigger, and K is the price which the issuer is obliged to purchase the stock when the loss exceeds D.

3.2 Loss Claims Process

If a catastrophe event occurs, any insurance company which has experienced loss whose share price will also show a down jump. Therefore, it is quite possible that the put options will be in-the-money, and reinsurance companies will be required to purchase shares at an unfavorable price. It is desired to have a model that describes the stock value process and the dynamic changes of the loss. Cox et al. (2004) introduced a model for the first time to price options for catastrophe association. They assumed that the asset price process is dominated by geometric Brownian motion and jumps down a predetermined size in the event of a catastrophe event. They assumed that only catastrophe events would affect the price of the stock, and the size of the catastrophe itself is uncorrelated to the stock price.

In this thesis, NHPP models are used when the occurrences of events are time-related. The intensity function $\{\lambda_t, t \geq 0\}$ of a NHPP $\{N_t, t \geq 0\}$ is a function of time. The most well-known definition of NHPP is described as follows.

(1) $N_0(0) = 0$

(2) $\{N_t, t \geq 0\}$ has independent increments.

(3) $P(N_{t+h} - N_t = 1) = \lambda_t h + o(h)$

(4) $P(N_{t+h} - N_t \geq 2) = o(h)$.

The main feature of a NHPP is that $N_{t+s} - N_t$ follows a Poisson distribution with the mean value function $m_{t+s} - m_t$, where $m_t = E[N_t]$, and it can be expressed as the integration of the intensity function

$$m_t = \int_0^t \lambda_u du.$$

3.3 Accumulated Loss Distribution

The accumulated insured property losses L_t is given by

$$L_t = \sum_{j=1}^{N_t} X_j,$$

That is, L_t is characterized by a compound process with two main components: one characterizing the frequency (or incidence) of catastrophic events and another describing the severity (or size or amount) of gain or loss resulting from the occurrence of a catastrophic event (Klugman et al., 2008; Tse, 2009). The occurrences of potentially catastrophic events specified in the contract is defined in terms of an adapted process $\{N_t, t \geq 0\}$. This adapted process is assumed to be a NHPP representing, for example, the number of hurricanes during [0,*t*). The underlying NHPP is described by its intensity function λ_t and mean value function $m_t = \int_0^t \lambda_u du$. The insured losses incurred by each event in the time flow $0 \leq t_1 \leq t_2 \leq \cdots \leq T$ are assumed to be independently and identically distributed (iid) random variables $\{X_j\}_{j\geq 1}$ with the distribution function $G(x) = P(X \leq x)$ and the probability density function (pdf) $g(x) = dG(x)/dx$. Random variables N_t and $\{X_j\}$ are assumed independent.

The distribution function of L_t is given by

$$P(L_t \leq x) = \sum_{n=0}^{\infty} G^{(n)}(x) \frac{[m_t]^n}{n!} e^{-m_t},$$

where $G^{(n)}(x) = P(X_1 + \cdots + X_n \leq x)$ is the *n*-fold convolution of *G*. It is known that the expectation of L_t is given by

$$EL_t = E(N_t)E(X_k) = m_t \cdot w$$

where $\omega = E(X_k)$

Let $Y = \inf\{t: L_t \geq D\}$ denote the time when the total number of claims exceeds the threshold level D for the first time. The expectation of Y is given by

$$E(Y) = \int_0^\infty P(Y > t)dt = \sum_{n=0}^\infty G^{(n)}(D) \int_0^\infty \frac{[m_t]^n}{n!} e^{-m_t} dt,$$

where we use the fact that $P(Y > t) = P(L_t < D)$. When $\{N_t, t \geq 0\}$ is a HPP with parameter λ, the expectation of Y can be approximated by renewal theory as

$$E(Y) = \sum_{n=0}^\infty G^{(n)}(D) \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} dt = \frac{1}{\lambda} \sum_{n=0}^\infty G^{(n)}(D) \approx \frac{1}{\lambda} \left(\frac{D}{\omega} + \frac{{\sigma_G}^2 + \omega^2}{2\omega^2}\right)$$

where $\omega = E(X_k)$ and ${\sigma_G}^2 = \mathrm{Var}(X_k)$.

4. Pricing Double Trigger Put Option

In this section, the SDE and the valuation of double trigger put option are given.

4.1 SDE

The price per share *S* of the double trigger put option is defined by the stochastic equation

$$S_t = S_0 \exp\left\{-\alpha L_t + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right\}$$

where S_0 is the initial price, μ is the mean return of the asset, and σ is the volatility of return. The factor $\alpha \geq 0$ gives the percentage drop in the share value per unit of loss (or a measure of impact of the level of claims on the market price of the share value). Let $\{W_t : t \geq 0\}$ be a standard Brownian motion, and $\{L_t : t \geq 0\}$ be an accumulated loss process. We assume that these two stochastic processes are independent.

This SDE is a generalization of Black-Scholes formula in the sense that if no large loss claims occur during the interval $(t, t+\Delta t)$ then the price S_t follows a geometric Brownian motion. Otherwise, the price changes from S_{t-} to $S_t = e^{-\alpha y} S_{t-}$, where *y* is the amount of loss claimed during the interval $(t, t+\Delta t)$

Let r be a risk-free interest rate. Let *Q* denote the new measure induced by the change of processes. Define a new Brownian motion under *Q*

$$\tilde{W}_t = \frac{\mu - k - r}{\sigma} t + W_t.$$

Thus, we have *Q*-a.s.,

$$S_t = S_0 \exp\{-(\alpha L_t - kt)\} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right\}.$$

Measure Q is equivalent to measure P and

$$\frac{dQ}{dP} = \exp\left[-\frac{\mu - k - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - k - r}{\sigma}\right)^2 T\right]$$

The equilibrium price measure Q exists and it is finite. The discounted relative price processes $\{e^{-rt}S_t : t \geq 0\}$ is a Q-martingale.

As argued by Cox et al. (2004) and Jaimungal and Wang (2005), the factor k is chosen such that $E^P\left[e^{-\alpha L_t + kt}\right] = 1$. It can be shown that

$$k = \frac{m_t}{t}\int_0^\infty (1 - \exp(-\alpha y))g(y)dy \,.$$

4.2 Catastrophe Put Option

Under the risk-neutral pricing measure Q, the value of double trigger put option depends on two contingencies, catastrophe events and option prices:

$$I_{\{L_T - L_t > D\}}(K - S_T)_+ = \begin{cases} K - S_T, & S_T < K \;\; and \;\; L_T - L_t > D \\ 0, & S_T \geq K \;\; or \;\; L_T - L_t \leq D \end{cases}$$

where $0 \leq t \leq T$.

Let V_t denote the value of the option at time t, then the value of double trigger put option at time t can be expressed via discounted expectations:

$$V_t = E_t^Q[e^{-r(T-t)} 1_{\{L_T - L_t > D\}}(K - S_T)_+ \mid F_t]$$

where E_t^Q denotes expectation under the risk-neutral pricing measure Q, given F_t.

The value of V_t can be obtained by conditioning on L_T and taking expectation:

$$V_t = E_t^Q[e^{-r(T-t)} 1_{\{L_T - L_t > D\}} E_t^Q[(K - S_T)_+ \mid L_T] \mid F_t]$$

Thus,

$$V_t = \int_{D^*}^{\infty} [e^{-r(T-t)} E_t^Q[(K - S_T)_+ \mid L_T = y \mid F_t]] dP(L_T \leq y)$$

The price $e^{-r(T-t)}E_t^Q[(K-S_T)_+ \mid L_T = y]$, conditionally on $L_T = y$, is given by the usual Black-Scholes formula, with an initial starting price of $S_t \exp\{-(\alpha L_t - kt)\}$, for $L_T > D^*$ with $D^* = \max(D - L_t, 0)$. Thus we have

$$e^{-r(T-t)}E_t^Q[(K-S_T)_+ \mid L_T = y] = Ke^{-r(T-t)}\Phi(-d_1(y) + \sigma\sqrt{T-t}) - S_0 e^{-\alpha y + k(T-t)}\Phi(-d_1(y))$$

where

$$d_1(y) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) - (\alpha y - k(T-t))}{\sigma\sqrt{T-t}}$$

Recall that,

$$dP(L_T \le y) = \sum_{n=0}^{\infty} g^{(n)}(y)\frac{[m_T]^n}{n!}e^{-m_T}$$

Therefore the value of the option at time *t* is given by

$$\begin{aligned} &V_t \\ &= E_t^Q[e^{-r(T-t)}1_{\{L_T - L_t > D\}}(K-S_T)_+ \mid F_t] \\ &= \sum_{n=0}^{\infty}\frac{[m_{(T-t)}]^n}{n!}e^{-m_{(T-t)}}\int_{D^*}^{\infty} g^{(n)}(y)[Ke^{-r(T-t)}\Phi(-d_1(y)+\sigma\sqrt{T-t}) - S_t e^{-\alpha y + k(T-t)}\Phi(-d_1(y))]dy \end{aligned}$$

where *K* is the exercise price, *r* is the interest rate, *T* is the term of the option, $L_T - L_t$ is the overall claims during $[t,T)$, $D^* = \max(D - L_t, 0)$, m_t is the expected number of large claims, S_t is the price of the underlying asset at time *t*, *k* is given by

$$k = \frac{m(t)}{t}\int_0^{\infty}(1-e^{-\alpha y})g(y)dy.,$$

and $d_1(y)$ is given by

$$d_1(y) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) - (\alpha y - k(T-t))}{\sigma\sqrt{T-t}}.$$

The loss trigger level *D* can be set as the multiple of the expected loses over the term of option:

$$D = cE[N_T - N_t]E[X_k] = c(m_T - m_t)\omega,$$

where c is a constant representing the *trigger ratio level*. For example, if c=2 is chosen, the trigger level is equal to twice of the product of the expected loss per claim and the expected number of claims during $[t,T)$.

The factor α gives the *percentage drop* (or impact) in the share value per unit of loss. This factor is usually taken to be 0.02 (Jaimungal and Wnag., 2005). When the occurrence of claims has no impact on the price of the underlying asset, i.e., $\alpha = 0$ (and consequently $k = 0$), the value of the option V_t becomes

$$V_t^0 = \sum_{n=0}^{\infty} \frac{[m_{(T-t)}]^n}{n!} e^{-m_{(T-t)}} \int_{D^*}^{\infty} g^{(n)}(y)[Ke^{-r(T-t)}\Phi(-d_1 + \sigma\sqrt{T-t}) - S_t\Phi(-d_1)]dy$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The pricing model V_t contains several models as special cases. They are:

Special Case 1: Model of Black and Scholes model (1973)

When the price of the underlying asset has no jump and follows the geometric Brownian motion:

$$S_t = S_0 \exp[\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma\widetilde{W_t}],$$

and the pricing of put option can be expressed as the usual Black-Schloes model:

$$V_t = Ke^{-r(T-t)}N(-d_1 + \sigma\sqrt{T-t}) - S_tN(-d_1).$$

Special Case 2: Model of Cox et al. (2004)

Define the payoff of catastrophe put option as $C_T = 1_{\{N_t \geq n\}}(K - S_T)_+$ and $D(0,T) = e^{-rT}$ as

$$1_{\{N_t \geq n\}}(K - S_T)_+ = \begin{cases} K - S_T & if\ S_T < K \text{ and } N_t \geq n \\ 0 & if S_T \geq K \text{ or } N_t < n \end{cases}$$

The pricing formula becomes

$$V_T = \sum_{j=n}^{\infty} [Ke^{-rT}\Phi\big(d_j\big) - S_t e^{-Aj+kT}\Phi(d_j - \sigma\sqrt{T})]e^{-\lambda T}\frac{(\lambda T)^j}{j!}$$

where $d_j = \frac{ln\frac{K}{S_0} - rT + Aj - kT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$ and A denotes the measurement of impact of level of claims.

Special Case 3: Model of Jaimungal and Wang (2005)

Let $L_t = \sum_{j=0}^{N_t} l_j$ denote the accumulated loss of the insured at time t, where $\{N_t, t \geq 0\}$ is the HPP claims arrival process and l_j is the amount of loss per claim. Let $D(0,T) = e^{-rT}$ and $C_T = 1_{\{L_T > D\}}(K - S_T)_+$. The pricing formula becomes

$$V_t = \sum_{j=1}^{\infty} \{\int_D^{\infty} f^{(j)}\,(y)[Ke^{-r(T-t)}\Phi\big(d(y)\big) - S_0 e^{-\alpha(y-kT)}\Phi(d_j - \sigma\sqrt{T})]e^{-\lambda T}\frac{(\lambda T)^j}{j!}\}$$

where $f(y)$ is the pdf of the loss distribution, and k satisfies

$$k = \frac{\lambda}{\alpha}\int_0^{\infty}(1 - e^{-\alpha y})\, f(y)dy.$$

5. Evaluation of the Pricing Model

The pricing model derived in the previous section is given by

$$V_t = \sum_{n=0}^{\infty} \frac{[m_{(T-t)}]^n}{n!} e^{-m_{(T-t)}} \int_{D^*}^{\infty} g^{(n)}(y)[Ke^{-r(T-t)}\Phi(-d_1(y)+\sigma\sqrt{T-t}) - S_t e^{-\alpha y+k(T-t)}\Phi(-d_1(y))]dy.$$

This pricing model involves the accumulated loss distribution, the NHPP, and the Black-Scholes model. In order to evaluate the value of the option, we indicate in this section three distributions for the size of claims and three NHPPs for loss claim arrivals. In next section, we use a real data set to fit both distributions and NHPPs in order to calibrate parameters of the pricing model.

5.1 Approximation for Accumulated Loss Distribution

The pricing model V_t involves the accumulated loss distribution, which is difficult to compute especially when n is large. To cope with this difficulty, some well-known methods for the accumulated loss distribution $F_L(y) = P(L_t \le y)$ are developed: the Fast Fourier Transform, simulation method, approximation method, etc. This thesis uses an approximation method to evaluate the accumulated loss density function $f_L(y) = P(L_t = y) = \sum_{n=0}^{\infty} \frac{[m_{(T-t)}]^n}{n!} e^{-m_{(T-t)}} g^{(n)}(y)$.

The (re-)insurance industry has suffered huge losses due to catastrophe risk events. This brings the application of the so-called heavy-tailed distributions for large size of loss claims. The heavy-tailed distributions include the log-normal, the log-gamma, and the Weibull distributions, among others (see Mikosch, 2004, Chapter 3). The use of gamma distribution is also popular in pricing catastrophe risk derivatives (Jaimungal and Wang, 2005; Ma and Ma, 2013).

In general, the approximation of $f_L(y)$ is $h_L(y)$, which is a function that uses the mean $\mu_h = E[L_t]$, the variance $\sigma_h^2 = Var[L_t]$, and even the skewness

$\alpha_3 = \mu_{3h} / \sigma_h^3$ and the excess kurtosis $\alpha_4 - 3 = (\mu_{4h} / \sigma_h^4) - 3$, where μ_{kh} is the kth moment about μ_h. Following the same approach by Lee and Yu (2002) and references therein, we only need to set the first two moments of $h_L(y)$ to be equal to $f_L(y)$. In this thesis, we consider gamma, log-normal, and Weibull distributions as the loss claims distributions.

(A) The density function of a gamma distribution X with shape parameter α and scale parameter β is given by

$$g(x;\alpha,\beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, \qquad x > 0, \alpha > 0, \beta > 0.$$

The mean and variance of X are, respectively,

$$E(X) = \alpha\beta$$

$$Var(X) = \alpha\beta^2$$

The accumulated gamma loss distribution $f_L(y)$ can be approximated by $h_L(y)$, which is a gamma distribution with mean μ_h and variance σ_h^2 given by, respectively,

$$\mu_h = E(L_t) = E[N(t)]E[X] = m(t) \cdot \alpha\beta$$

$$\sigma_h^2 = Var(L_t) = E[N(t)]E[X^2] = m(t) \cdot (\alpha\beta^2 + \alpha^2\beta^2).$$

(B) The density function of a log-normal distribution X with location parameter μ and scale parameter σ is given by

$$g(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2} \right), \qquad x > 0, \sigma > 0, -\infty < \mu < \infty.$$

The mean and variance of X are, respectively,

$$E(X) = \exp\left(\mu + \frac{\sigma^2}{2} \right)$$

$$Var(X) = \exp\left(2\mu + 2\sigma^2\right) - \exp\left(2\mu + \sigma^2\right).$$

The accumulated log-normal loss distribution $f_L(y)$ can be approximated by $h_L(y)$, which is a log-normal distribution with mean μ_h and variance σ_h^2 given by, respectively,

$$\mu_h = E(L_t) = E[N(t)]E[X] = m(t)\cdot \exp(\mu + \tfrac{1}{2}\sigma^2)$$

$$\sigma_h^2 = Var(L_t) = E[N(t)]E[X^2] = m(t)\cdot \exp(2\mu + 2\sigma^2)$$

(C) The density function of a Weibull distribution *X* with shape parameter γ and scale parameter β is given by

$$g(x;\gamma,\beta) = \frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1} \exp\left\{-\left(\frac{x}{\beta}\right)^{\gamma}\right\}, \qquad x>0, \beta>0, \gamma>0$$

The mean and variance of *X* are, respectively,

$$E(X) = \beta\cdot\Gamma\left(1+\frac{1}{\gamma}\right)$$

$$Var(X) = \beta^2\cdot\left[\Gamma\left(1+\frac{2}{\gamma}\right) - \Gamma^2\left(1+\frac{1}{\gamma}\right)\right].$$

Following the same approach for the log-normal distribution, the accumulated Weibull loss distribution $f_L(y)$ is approximated by $h_L(y)$, which is a Weibull distribution with mean μ_h and variance σ_h^2 given by , respectively,

$$\mu_h = E(L_t) = E[N(t)]E[X] = m(t)\cdot\beta\cdot\Gamma\left(1+\frac{1}{\gamma}\right)$$

$$\sigma_h^2 = Var(L_t) = E[N(t)]E[X^2] = m(t)\cdot\beta^2\cdot\Gamma\left(1+\frac{2}{\gamma}\right).$$

5.2 Non-homogeneous Poisson Process

NHPP is more appropriate than HPP when events are more likely to occur at certain times than at the other times. Since the loss claims can arrive in cluster, thus it may be suitable to assume that the catastrophe loss claims arrive according to a NHPP.

One important feature of NHPP is the property of independent increments. One well-known method for constructing NHPP is to solve the following differential equation:

$$\frac{dm(t)}{dt} = b(t)[a(t) - m(t)]$$

where $m(t)$ is the expected number of loss claims occurred by time t, $a(t)$ is the total number of observed loss claims by time t, and $b(t)$ is the occurrence rate per claim at time t. Three NHPP models are given in Table 1. They are the Goel-Okumoto model, the delayed S-shaped model, and the inflection S-shaped model (Lyu, 1996).

Model Name	$m(t)$	$\lambda(t)$
Goel-Okumoto	$m(t) = a(1 - e^{-bt})$ $a(t) = a$ $b(t) = b$	$\lambda(t) = abe^{-bt}$
Inflection S-shaped	$m(t) = \dfrac{a(1 - e^{-bt})}{1 + \beta e^{-bt}}$ $a(t) = a$ $b(t) = \dfrac{b}{1 + \beta e^{-bt}}$	$\lambda(t) = \dfrac{abe^{-bt}(1 + \beta)}{(1 + \beta e^{-bt})^2}$
Delay S-shaped	$m(t) = a[1 - (1 + bt)e^{-bt}]$ $a(t) = a$ $b(t) = \dfrac{b^2 t}{1 + bt}$	$\lambda(t) = ab^2 te^{-bt}$

Table 1. NHPP models.

6. Parameters Calibration

To estimate and calibrate the parameters of the pricing model, we need to fit $h_L(y)$, the approximated density function of accumulated loss distribution, and N_t, the loss claims process.

6.1 Data Set

The generic data set of Munich Re's NatCatSERVICE is a database for evaluating and analyzing natural catastrophes. The institution has recorded the loss data from all over the world since 1980. It only records events whose losses are over a predetermined threshold. The database provides the information that could be applied for assessing the risk. The insurance loss and total loss of natural catastrophe events are listed individually in each form. For example, ten costliest Hurricanes in the US 1980-2015 are shown in Table 2.

Time	**Event**	**Overall Losses in US$ m**	**Insured Losses in US$ m**
Aug, 2005	Hurricane Katrina	125000	60500
Nov, 2012	Hurricane Sandy	68500	29500
Sep, 2008	Hurricane Ike	38000	18500
Aug, 1992	Hurricane Andrew	26500	17000
Sep, 2004	Hurricane Ivan	23000	11800
Oct, 2005	Hurricane Wilma	22000	12500
Aug, 2004	Hurricane Charley	18000	8000
Sep, 2005	Hurricane Rita	16000	8600
Sep, 1998	Hurricane Georges	13300	4300
Sep, 2004	Hurricane Frances	12000	5500

Table 2. Ten Costliest Hurricanes in United States 1980-2015.

To validate the pricing model, we apply the data set used by Ma and Ma (2013). This data set is a part of the Property Claim Service (PCS) loss data for catastrophe events occurred in the US from 1985 to 2010. The amount of losses is transformed to 2010 dollars using the Consumer Price Index (CPI) to adjust for inflation. The adjusted PCS loss data between 1985 and 2010 is shown in Figure 2. The cumulative number of catastrophe events in the US between 1985 and 2010 is given in Figure 3.

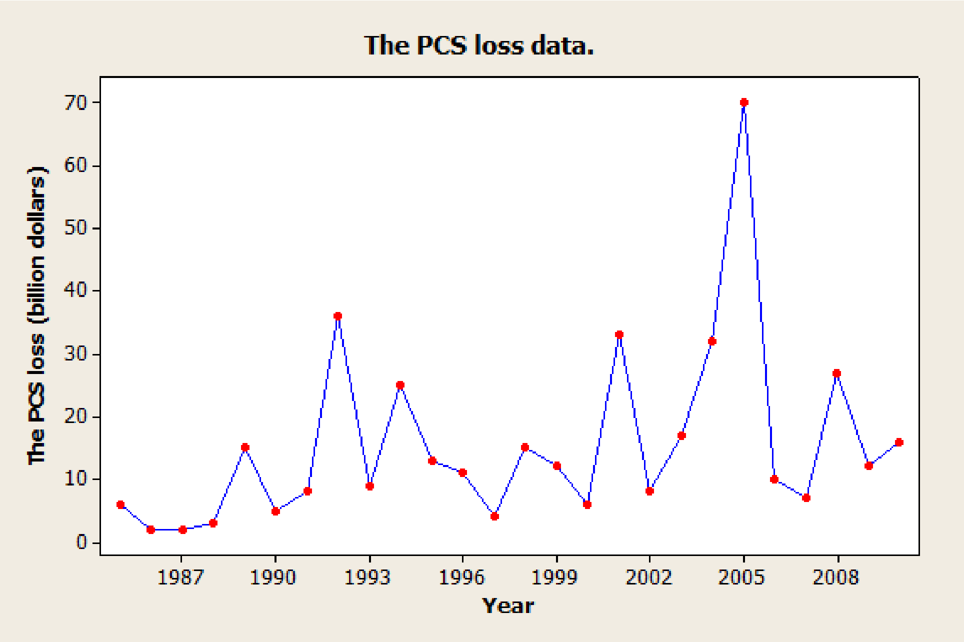

Figure 2. The PCS loss (billion dollars).

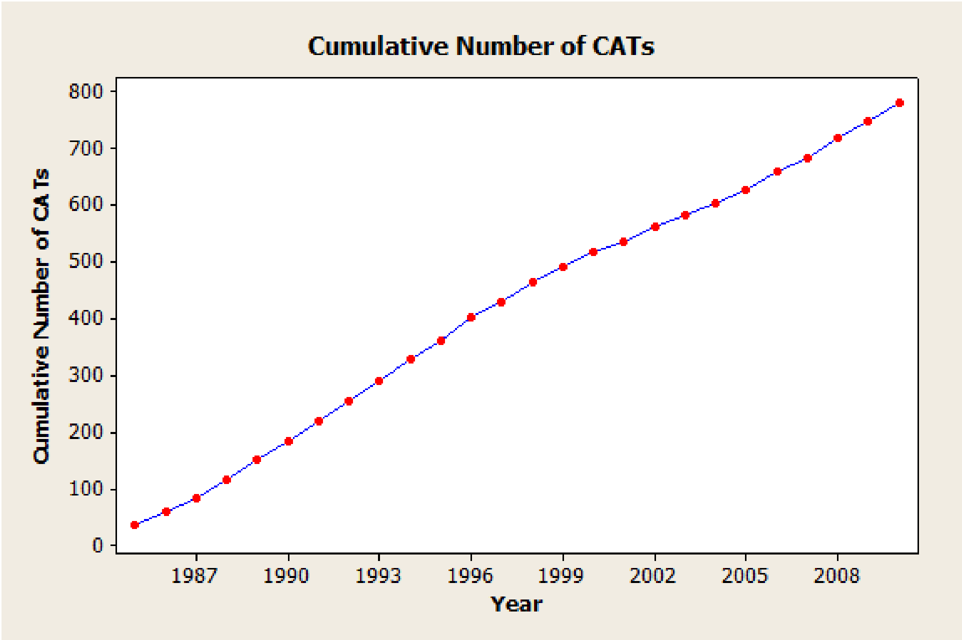

Figure 3. Cumulative number of CATs.

6.2 Estimation of Accumulated Loss Distribution

Three loss claims distributions are the gamma, the log-normal, and the Weibull distributions. Parameters of the distribution functions are estimated by maximum likelihood estimation (MLE). The results of the parameter estimation are given in Table 3.

The accumulated loss distribution is one part of the pricing model, and it affects the option prices. To evaluate the effectiveness of fitting performance, a set of comparison criteria is employed to compare chosen distributions quantitatively. The comparison criteria we used are AD statistic and AIC.

Table 3 shows that all three distributions pass the AD statistic test at 5% level of significance. The log-normal with parameters $\mu = 2.382$ and $\sigma = 0.8844$, and the gamma distributions with parameters $a = 1.529$ and $b = 10.16$ have lower AD values, we thus use both distributions as accumulated loss amount for numerical analysis in next section. Also, Table 3 shows AIC values of all distributions.

(A) *Anderson-Darling (AD) statistic*: The AD test is a form of minimum distance estimate, which is designed to detect the difference in the tails between the data and the fitted distribution. The AD statistic is defined by

$$\mathrm{AD} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\cdot\left[\ln F(X_i) + \ln(1-F(X_{n-i+1}))\right] - n.$$

If the test statistic AD is greater than the corresponding critical value C_α (at α level of significance), we reject the data follow the predetermined distribution.

(B) *Akaike information criterion (AIC)*: AIC is computed as follows (Konish and Kitagawa, 2008):

$$\mathrm{AIC} = -2\ln(L) + 2k,$$

where k is the number of parameters in the model, and L is the maximized value of the likelihood function for the estimated model. AIC takes the degrees of freedom into consideration by assigning a model with more parameters a larger penalty. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best.

Distribution	Log-normal	gamma	Weibull
Parameters	$\mu = 2.382$ $\sigma = 0.8844$	$a = 1.529$ $b = 10.16$	$\gamma = 1.206$ $\beta = 16.65$
AD ($C_{0.05} = 2.5018$)	**0.161**	**0.315**	0.373
AIC	214.1464	198.7547	199.8861

Table 3. Parameter estimates for accumulated loss distributions.

6.3 Estimation of Loss Claims Process

Parameters of the loss claims process are estimated by the least square errors (for the Inflection S-shaped model only) and the maximum likelihood procedure. From Table 4, the Goel-Okumoto model with parameters $a = 2935$ and $b = 8.0286 \times 10^{-3}$, and the Delayed S-shaped model with parameters $a = 982$ and $b = 0.1137$ have lower values of AIC. Thus we use the Goel-Okumoto and the Delayed S-shaped models for loss claims processes.

NHPP Processes	Inflection S-shaped	Delayed S-shaped	Goel-Okumoto
Parameters	$a = 1101$ $b = 7.445 \times 10^{-5}$ $\beta = 1.668 \times 10^{-3}$	$a = 982$ $b = 0.1137$	$a = 4140$ $b = 8.0286 \times 10^{-3}$
AIC	5630	**258**	**169**

Table 4. Parameter estimates for loss claims processes

Note that parameter a is the expected number of events that initially exist in the flow of occurrences, and parameter b is the occurrence rate per event. For the Inflection S-shaped model, parameter β is equal to $(1-r)/r$ where r is the inflection factor (Lyu, 1996).

7. Numerical Analysis for the Pricing Model

In this section, the pricing model V_t is analyzed. We will discuss influences of the choice of the pricing model parameters. Recall that the pricing model is given by

$$V_t = \sum_{n=0}^{\infty} \frac{[m_{(T-t)}]^n}{n!} e^{-m_{(T-t)}} \int_{D^*}^{\infty} g^{(n)}(y)[Ke^{-r(T-t)}\Phi(-d_1(y)+\sigma\sqrt{T-t}) - S_t e^{-\alpha y + k(T-t)}\Phi(-d_1(y))]dy$$

where

$$d_1(y) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) - (\alpha y - k(T-t))}{\sigma\sqrt{T-t}}$$

For the accumulated loss distributions, we consider the log-normal distribution with parameters $\mu = 2.382$ and $\sigma = 0.8844$, and the gamma distributions with parameters $a = 1.529$ and $b = 10.16$. For the NHPP loss claims arrivals, we consider the Goel-Okumoto model with parameters $a = 4140$ and $b = 8.0286 \times 10^{-3}$, and the Delayed S-shaped model with parameters $a = 982$ and $b = 0.1137$.

We assume that the maturity time $T \in [1,5]$ years, because an option usually has 1 to 5 year term. The loss trigger level D is assumed to be the multiple of the expected losses:

$$D \approx cE[L_T]$$

where c is a constant representing the trigger ratio level. For example, if c=1 is chosen, the trigger level is equal to the product of the expected loss per claim and the expected number of claims during [0,T]. We assume that the trigger ratio level is c=1.

The factor α gives the percentage drop in the share value per unit of loss, and the factor k is chosen such that

$$k = \frac{m_t}{t}\int_0^{\infty}(1-\exp(-\alpha y))g(y)dy.$$

The value of factor k is determined by the value of factor α. The following table shows values of k for maturity time $T \in [1,5]$ when value of α is specified under the Goel-Okumoto NHPP and the gamma loss assumptions.

	T=1	T=2	T=3	T=4	T=5
$\alpha = 0$	0	0	0	0	0
$\alpha = 0.001$	0.015477	0.015415	0.015353	0.015292	0.015231
$\alpha = 0.01$	0.149781	0.149182	0.148586	0.147994	0.147404
$\alpha = 0.02$	0.289401	0.288244	0.287093	0.285948	0.284809
$\alpha = 0.05$	0.658817	0.656183	0.653563	0.650957	0.648365
$\alpha = 0.1$	1.155368	1.150748	1.146154	1.141583	1.137037
$\alpha = 0.5$	3.166444	3.153783	3.141191	3.128665	3.116206

Table 5. Values of *k* for different drop percentage α and maturity time *T* (year) using Goel-Okumoto NHPP and gamma loss.

Parameters are set by: exercise price K=80; interest rate, r=0.05; volatility, $\sigma = 0.2$; percentage drop, $\alpha = 0.02$; maturity time, $T \in [1,5]$ years; price of the underlying asset, $S \in [0,110]$. We now calculate the option price for various models.

Figure 4 shows the catastrophe risk option price with respect to the price of asset and maturity time under Goel-Okumoto NHPP and gamma loss assumptions. Figure 5 shows the catastrophe risk option price with respect to the price of underlying asset and maturity time under Goel-Okumoto NHPP and log-normal loss assumptions. Both figures show that the option prices decrease as the price of asset increases, and increase as the time to maturity increases (for asset prices S larger than 60). For small asset prices ($S < 60$), the option prices have maximum values around maturity $T = 2$ or $T = 3$.

Figure 6 shows the catastrophe risk option price with respect to the price of asset and maturity time under Delay S-shaped NHPP and gamma loss assumptions. The option prices decrease as the price of asset increases, and increase as the time to maturity increases (for asset prices S larger than 60). For small asset prices ($S < 60$), the option prices have maximum values around maturity $T = 1$.

Figure 7 shows the catastrophe risk option price with respect to the price of underlying asset and maturity time under Delay S-shaped NHPP and log-normal loss assumptions. The option prices decrease as the price of asset increases, and increase then decrease as the time to maturity increases. For a fixed asset price, the option prices reach their maximum values around maturity $T = 4$.

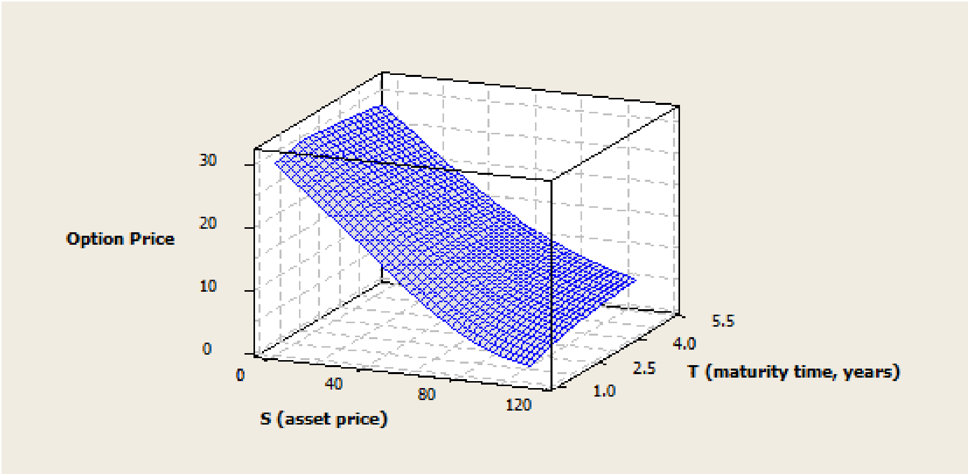

Figure 4. Catastrophe option price with Goel-Okumoto NHPP and gamma loss for *c*=1, *K*=80, *r*=0.05, $\sigma = 0.2$**,** $\alpha = 0.02$**.**

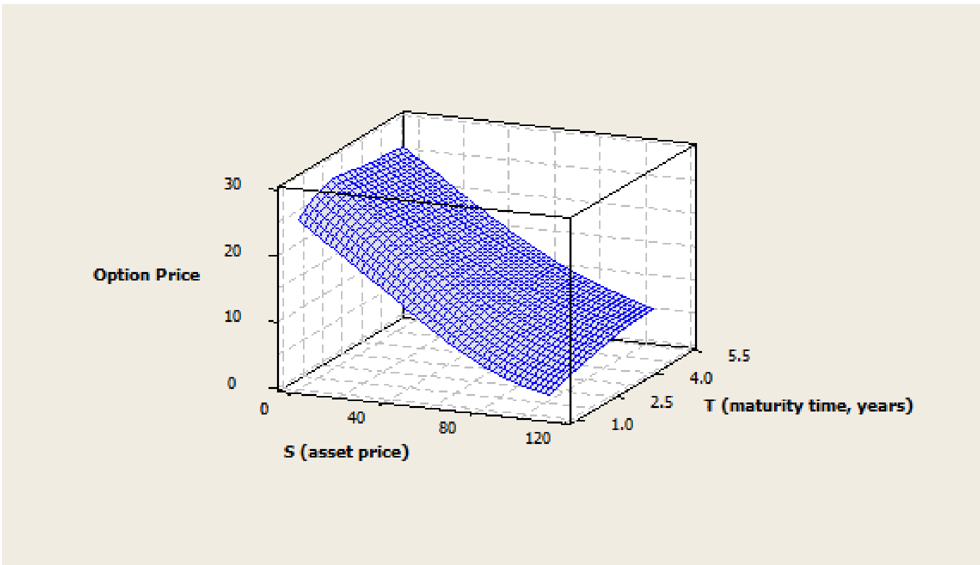

Figure 5. Catastrophe option price with Goel-Okumoto NHPP and log-normal loss for *c*=1, *K*=80, *r*=0.05, $\sigma = 0.2$**,** $\alpha = 0.02$**.**

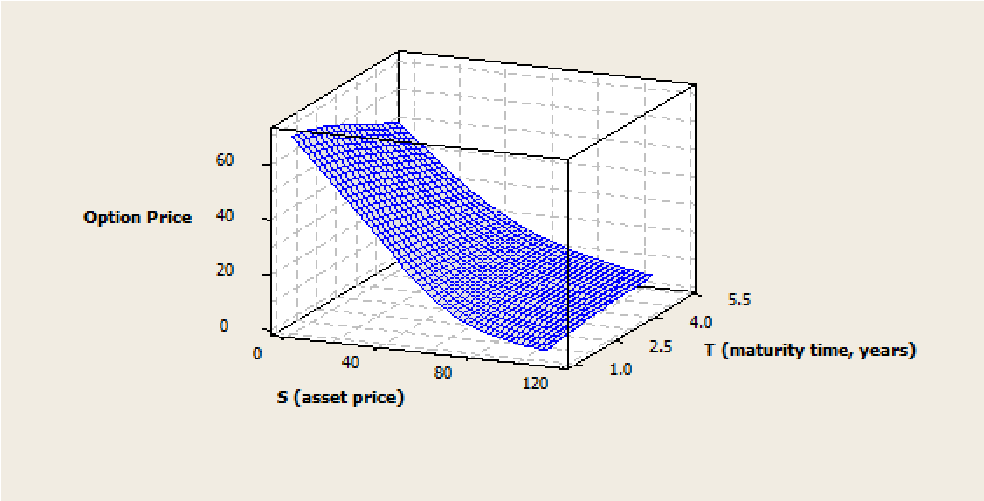

Figure 6. Catastrophe option price with Delay S-shaped NHPP and gamma loss for *c*=1, *K*=80, *r*=0.05, $\sigma = 0.2$, $\alpha = 0.02$.

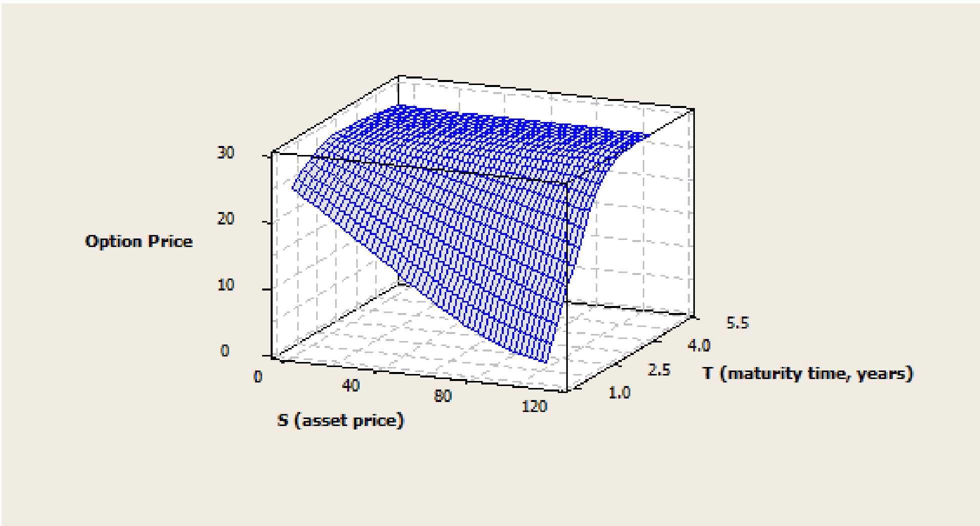

Figure 7. Catastrophe option price with Delay S-shaped NHPP and log-normal loss for *c*=1, *K*=80, *r*=0.05, $\sigma = 0.2$, $\alpha = 0.02$.

Figure 8 to Figure 13 show the price difference between trigger ratio level ($c = 1/4$ and $c = 1$), percentage drop ($\alpha = 0.02$ and $\alpha = 0$), loss distributions (gamma and log-normal), and NHPPs (Goel-Okumoto and Delay S-shaped).

Figure 8 shows the price difference between threshold trigger ratios $c = 1/4$ and $c = 1$ under Goel-Okumoto NHPP and gamma loss assumptions. We observe that trigger ratio level $c = 1/4$ overestimates the option prices. The option prices for a small trigger ratio level ($c = 1/4$) are higher than that for a large trigger ratio level ($c = 1$). The price difference increases sharply when the asset price is small. The significant price differences show that the trigger ratio level is an important factor and should be taken into account when pricing catastrophe risk option.

Figure 9 shows the price difference between percentage drop $\alpha = 0.02$ and $\alpha = 0$ under Goel-Okumoto NHPP and gamma loss assumptions. The option prices for $\alpha = 0.02$ are higher than those for $\alpha = 0$. The difference increase as the maturity time increases. For a fixed maturity time, the difference has a unimodal shape along the asset price S and reach its maximum value around $S = 60$. The price differences indicate that loss claims affect the option prices substantially. The option prices will increase (decrease) if the percentage drop α increases (decreases). Thus the percentage drop is an important factor and need to be taken much care.

Figures 10 and 11 show how the choice of loss claims distribution affect the option prices. Figure 10 illustrates the price difference between gamma and log-normal distributions under Goel-Okumoto NHPP assumption. The differences of the option prices vary from -0.767 to 4.822 dollars. Figure 11 illustrates the price difference between gamma and log-normal distributions under Delay S-shaped NHPP assumption. The differences of the option prices vary from -21.83 to 44.54 dollars. Both figures show that for small asset price (Goel-Okumoto: $S < 70$; Delay S-shaped: $S < 40$) the gamma loss distribution has higher option prices, while for large asset price the log-normal loss distribution has higher option price.

Figures 12 and 13 show how the choice of NHPP affect the option prices. Figure 12 illustrates the price difference between Goel-Okumoto and Delay S-shaped NHPPs under gamma loss assumption. The differences of the option prices vary from 0.0406 to 39.72 dollars. Figure 13 illustrates the price difference between Goel-Okumoto and Delay S-shaped NHPPs under log-normal loss assumption. The differences of the option prices vary from -21.577 to 0 dollars. Two NHPPs have opposite results under different distributions. The Goel-Okumoto NHPP overestimates the option prices under gamma loss assumption, while the Delay S-shaped NHPP overestimates the option prices under log-normal loss assumption. It is therefore need to pay attentions to select a suitable NHPP when pricing catastrophe risk option.

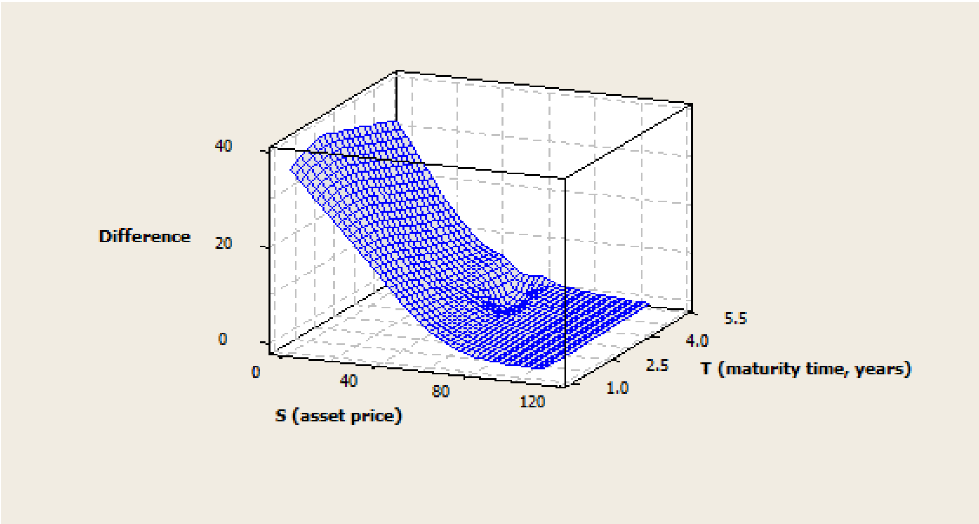

Figure 8. The price difference between threshold trigger ratios c=1/4 and c=1 under Goel-Okumoto NHPP and gamma loss assumptions.

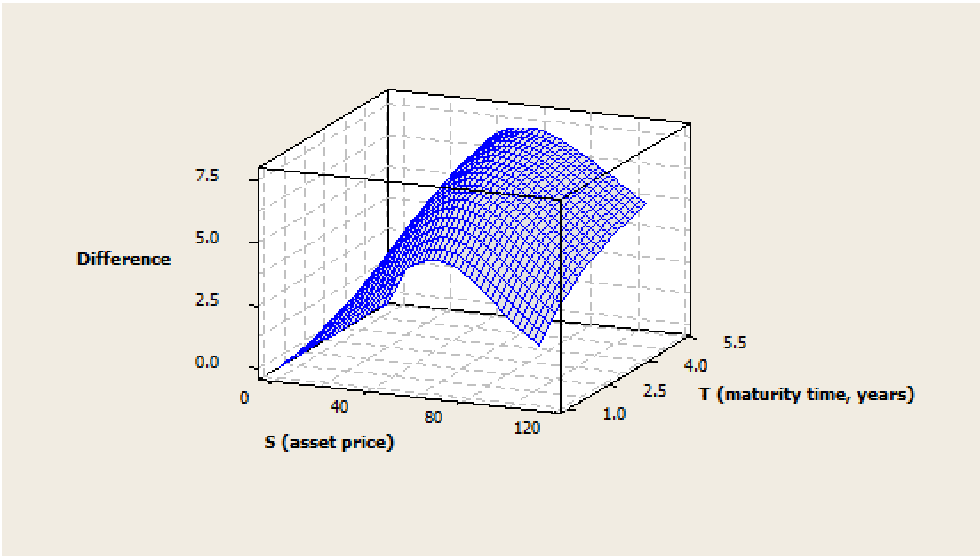

Figure 9. The price difference between $\alpha = 0.02$ and $\alpha = 0$ under Goel-Okumoto NHPP and gamma loss assumptions.

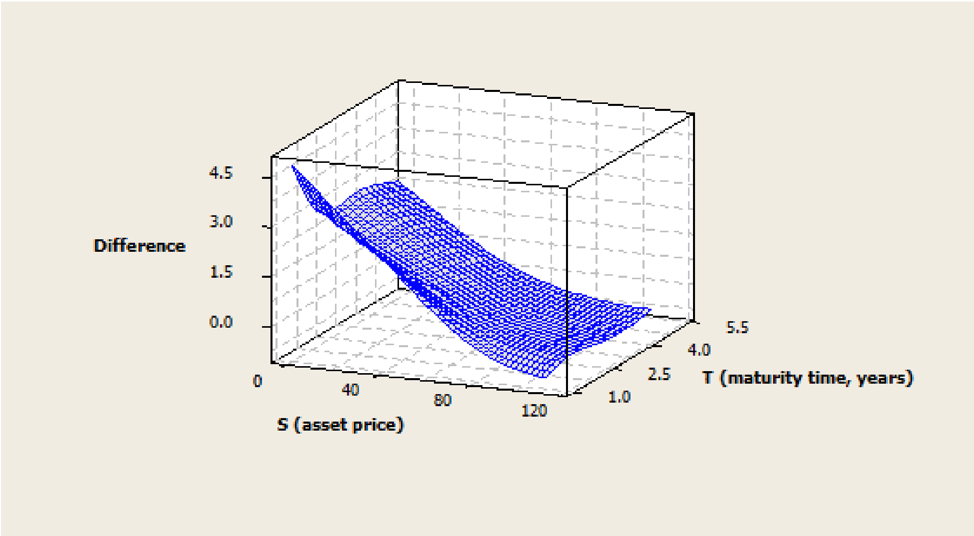

Figure 10. The price difference between gamma and log-normal loss distributions under Goel-Okumoto NHPP assumption.

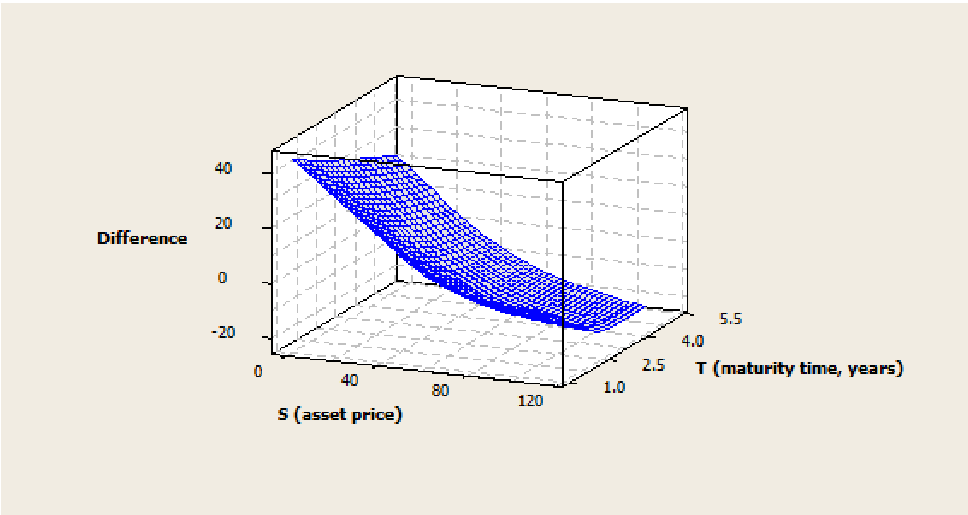

Figure 11. The price difference between gamma and log-normal loss distributions under Delay S-shaped NHPP assumption.

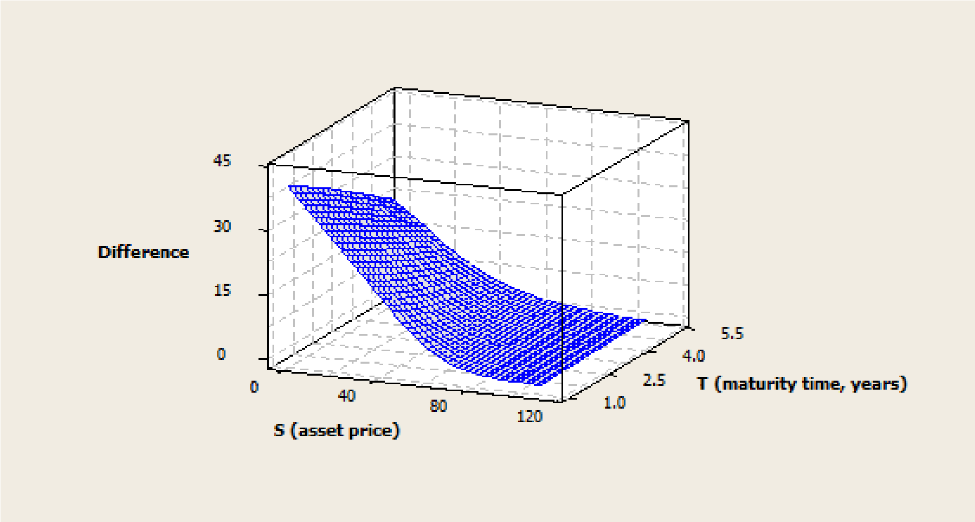

Figure 12. The price difference between Goel-Okumoto and Delay S-shaped NHPPs under gamma loss assumption.

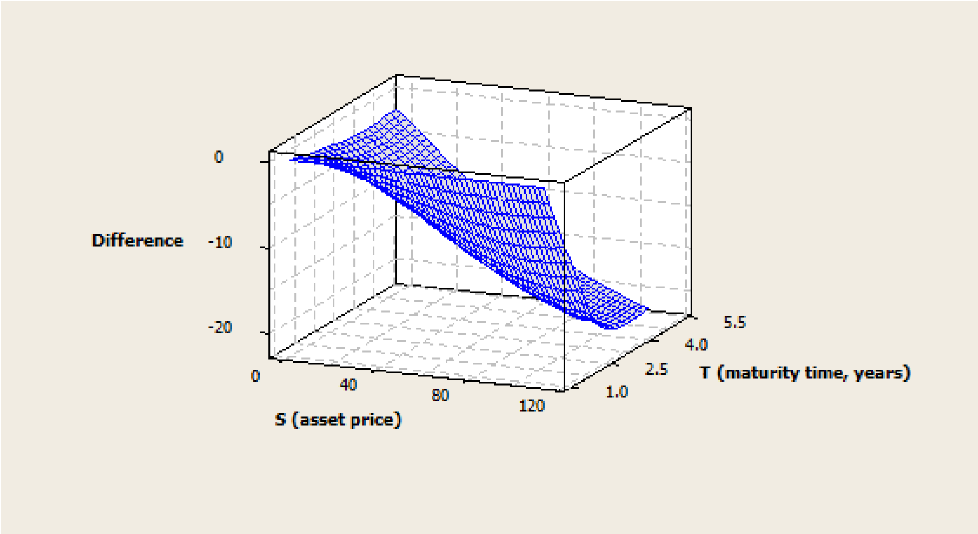

Figure 13. The price difference between Goel-Okumoto and Delay S-shaped NHPPs under log-normal loss assumption.

8. Conclusions

This thesis develops a model to price the catastrophe risk put option. The pricing model is developed by a double trigger put option, which depends on the underlying asset price and the cumulative level of insurance loss. We apply the valuation theory to derive the pricing model. Under the risk-neutral pricing measure, the value of double trigger put option is expressed via a discounted expectation, and the closed form of the pricing model is provided. This pricing model involves the accumulated loss distribution, the loss claims arrival process, and the Black-Scholes model. The accumulated loss distribution is characterized by a compound distribution and is approximated by heavy-tailed distributions. The loss claims arrivals process is assumed to be a non-homogeneous Poisson jump process.

We apply a real data set to fit both accumulated loss distributions and NHPP arrivals in order to calibrate parameters of the proposed pricing model. The numerical results show that trigger ratio level, percentage drop, loss distributions, and NHPPs have important influences for the pricing model. By comparing different trigger ratio levels, we observed that a small trigger ratio level has higher option prices than those of a large trigger ratio level. As the percentage drop α increase, the option price also increases. For the choice of loss distributions, the numerical results show that for small asset price S the gamma loss distribution has higher option prices. While for large asset price S the log-normal loss distribution has higher option prices. As for the choice of NHPPs, we validate that the Goel-Okumoto NHPP overestimates the option prices under gamma loss assumption, and the Delay S-shaped NHPP overestimates the option prices under the log-normal loss assumption.

Appendix A: Black-Scholes Model

It is well known that the process of price movement of derivatives can be derived from Ito's lemma, which was discovered by Ito in the1950s. The Ito's Lemma assume that the random variable *x* can be derived from Ito process

$$dx = a(x,t)dt + b(x,t)dW$$

where both *a* and *b* are the functions of *x* and *t*, dW is a Weiner process. Let *x* be the price of derivative. A function *f* of *x* and *t* can be expressed as

$$df(x,t) = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2\right)dt + \frac{\partial f}{\partial x}bdW$$

In the following, the Black-Scholes formula is derived from statistical point of view. Let

S_T: stock price at maturity time T

S_0: stock price at time 0

μ : asset return

σ : volatility.

Assume that

(1) The stock price follows log-normal distribution.

$$lnS_T \sim N\left(lnS_0 + \left(\mu - \frac{\sigma^2}{2}\right)T\ , \sigma^2 T\right)$$

Since lnS_T is normally distributed, such that S_T has a log-normal distribution.

(2) The short selling of securities with full use of proceeds is permitted.

(3) There are no dividends during the life of derivative.

(4) There are no transactions costs or taxes. All securities are perfectly divisible.

(5) There are no riskless arbitrage opportunities.

(6) Security trading is continuous.

(7) The risk-free rate of interest, *r*, is constant and the same for all maturities.

Pricing European call and put options

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The variables, c and p, are the price of European call and put options, S_0 is current stock price, r is the continuously compound risk-free rate, σ is the volatility of stock price, and T is the time to the maturity of the option.

Proof.

Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[max(S_T - K\,, 0)]$$

where $\hat{\mathrm{E}}$ denotes the expected value in a risk-neutral world. Assume that the expected return from the underlying asset is the risk-free interest rate, *r* (i.e., assume $\mu = r$).

Define $y \equiv ln\, S_T$, and *y* follows normal distribution. In a risk-neutral world,

$$y = lnS_T \sim N(lnS_0 + \left(r - \frac{\sigma^2}{2}\right)T\,, \sigma^2 T)$$

Let $m = lnS_0 + \left(r - \frac{\sigma^2}{2}\right)T$, $s^2 = \sigma^2 T$ and $f(S_T)$ be the probability density function of S_T.

$$\hat{E}[\max(S_T - K\,, 0)] = \int_0^\infty \max(S_T - K\,, 0)\, f(S_T) dS_T = \int_K^\infty (S_T - K)\, f(S_T) dS_T$$

$$= \int_{lnK}^\infty e^y \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{1}{2}\left(\frac{y-m}{s}\right)^2\right) dy - \int_{lnK}^\infty K \frac{1}{\sqrt{2\pi s^2}} exp\left(-\frac{1}{2}\left(\frac{y-m}{s}\right)^2\right) dy$$

$$= \int_{lnK}^\infty \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{1}{2s^2}(y^2 - 2my + m^2 - 2ys^2)\right) dy$$

$$- K \int_{lnK}^\infty \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{1}{2}\left(\frac{y-m}{s}\right)^2\right) d\, y$$

$$= \int_{lnK}^{\infty} \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{1}{2s^2}[(y-m-s^2)^2 + m^2 - (m+s^2)^2]\right) dy$$

$$- K \int_{\frac{lnK-m}{s}}^{\infty} \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{1}{2}\left(\frac{y-m}{s}\right)^2\right) d\left(\frac{y-m}{s}\right)$$

$$= \int_{\frac{lnK-m-s^2}{s}}^{\infty} \frac{1}{\sqrt{2\pi s^2}} \exp\left[-\frac{1}{2}\left(\frac{y-m-s}{s}\right)^2 - \frac{1}{2s^2}(-2ms^2 - s^4)\right] d\left(\frac{y-m-s^2}{s}\right)$$

$$- K\left[1 - N\left(\frac{lnK-m}{s}\right)\right] = e^{m+\frac{1}{2}s^2} N\left(\frac{m+s^2-lnK}{s}\right) - K \times N\left(\frac{m-lnK}{s}\right)$$

$$d_1 = \frac{m+s^2-lnK}{s} = \frac{lnS_0 + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma^2 T - lnK}{\sigma\sqrt{T}} = \frac{ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{m-lnK}{s} = \frac{m-lnK+s^2}{s} - s = d_1 - \sigma\sqrt{T}$$

$$e^{m+\frac{1}{2}s^2} = \exp\left[lnS_0 + \left(r - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T\right] = S_0 e^{rT}$$

Therefore, $\hat{E}[\max(S_T - K\,, 0)] = S_0 e^{rT} N(d_1) - KN(d_2)$ and the price of European call option

$$c = \hat{E}[\max(S_T - K\,, 0)]e^{-rT} = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Similarly, the price of European put option

$$p = \hat{E}[\max(K - S_T\,, 0)]e^{-rT} = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

References

Black, F., Scholes, M., 1973, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81(3): 637-654.

Cox, S. H., Fairchild, J. R., Pedersen, H. W., 2004, Valuation of structured risk management products, *Insurance: Mathematics and Economics*, 34: 259-272.

Cummins, J.D., 2008, CAT bounds and other risk-linked securities: state of the market and recent developments, *Risk Management and Insurance Review*, 11: 23-47.

Dassios, A., Jang, J. W., 2003, Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity, *Finance and Stochastics*, 7:93-95.

Dothan, M.U., 1990, *Prices in Financial Markets*, Oxford University Press.

D'Arcy, S.P., France, V.G., 1992, Catastrophe futures: a better hedge for insurers, *Journal of Risk and Insurance*, 59: 575-600.

Jaimungal, S., Wang, T., 2005, Catastrophe options with stochastic interest rates and compound Poisson losses, *Insurance: Mathematical and Economics*, 38(3):469-483.

Konishi, S., Kitagawa, G., 2008, *Information Criteria and Statistical Modeling*, Springer.

Klugman, S.A., Panjer, H.H., Willmot, G.E., 2008, *Loss Models: From Data to Decisions*, 3rd ed., Wiley, New York.

Laster, D.S., 2001, Capital market innovation in the insurance industry, *Sigma*, No.3, Zurich, Switzerland.

Lee, J.P., Yu, M.T., 2002, Pricing default-risky CAT bonds with moral hazard and basis risk, *Journal of Risk and Insurance*, 69(1): 25-44.

Lyu, M. R. (1996), *Handbook of software reliability engineering*, IEEE Computer Society Press, Los Alamitos.

Loubergé, H., Kellezi, E., Gilli, M., 1999, Using Catastrophe-Linked Securities to Diversify Insurance Risk: A Financial Analysis of CAT bonds, *Journal of Insurance Issues*, 22(2): 125-146.

Ma, Z. G., Ma, C. Q., 2013, Pricing catastrophe risk bonds: A mixed approximation method,*Insurance: Mathematics and Economics*, 52: 243-254.

Mikosch, T., 2004, Non-Life Insurance Mathematics, Springer, Berlin.

Punter, A., 2001, Securitisation of insurance risk, in: 2001 Hazard Conference, Aon Limited.

Tse, Y.-K., 2009, *Nonlife Actuarial Models: Theory, Models and Evaluation*, Cambridge University Press, Cambridge.